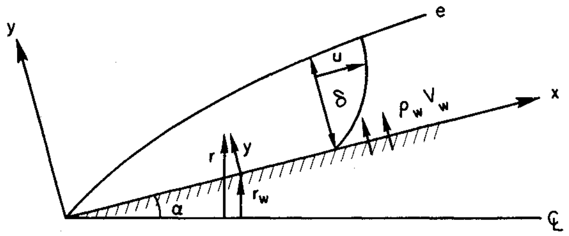

Fig. 1 Coordinate system.

It should be noted that Thyson and Schurmann[4] used a two-dimensional displacement thickness δ^*, and their formula does not fully account for the transverse curvature effect. In hypersonic boundary layers, $\delta^* \rightarrow \delta$; then, Eq. (10) becomes

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta^*}{dx} \tag{12}$$

Equation (12) provides the necessary matching condition used in Ref. 3.

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Reply by Authors to T.-Y. Li and J. F. Gross

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If transverse curvature effects are to be included, the inviscid streamline slope expression as given by Li and Gross is quite correct. In our original note, a transverse curvature effect was omitted which multiplies the rate of changes of the displacement thickness and inviscid flow. The slope is

$$\frac{V_e}{U_e} = \frac{\rho_w V_w}{\rho_e U_e}\left[\frac{1}{1+(\delta/r)}\right] + \frac{d\delta}{dx}\left\{1 - \left[\frac{1}{1+(\delta/r)}\right]\right\} + \left[\frac{d\delta^*}{dx} - \frac{\delta - \delta^*}{\rho_e U_e r}\frac{d}{dx}(\rho_e U_e r)\right]\left[\frac{1}{1+(\delta/r)}\right]$$

The difference then between the expression as given by Li and Gross and the foregoing expression is that they have properly included a transverse curvature effect in the axisymmetry displacement thickness. However, our displacement thickness is an axisymmetric displacement thickness. What we neglected was a transverse curvature effect on the displacement thickness. Since we did not consider transverse effects, our pressure interaction analysis as such is still valid.

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Comment on "Orbital Motion in the Theory of General Relativity"

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In their discussion[1] of relativistic orbital mechanics, Anderson and Lorell note the difficulty in interpreting the equations of motion. They go on to mention that, although their result for the secular change in the argument of pericenter agrees with that of Bogorodskii,[2] they do not match the latter in regard to his estimate of the effect on χ (i.e., $-n\tau$). The difficulty lies in the fact that, in the two papers, somewhat different perturbative components, R and S (the former radial, the latter tangential), are used. Further, Bogorodskii's error in computing changes in χ was not corrected, as is evident in Eq. (4) of Ref. 1.

That this equation is incomplete has been noted before,[3] although a very careful reading of the source material[4] is required in order to perceive this; other sources[5,6] are only slightly less ambiguous.

We find, employing the same notation as Anderson and Lorell, and following Brouwer and Clemence[6] (especially pages 285–286 and 300–301),

$$\overline{d\chi/dt} = [3\mu n/ac^2(1-e^2)] \times [2 + e^2 - (3 + 2e + e^2 + 2e^3)/(1-e^2)]$$

Incidentally, utilizing Bogorodskii's perturbations (but setting his ω_0 equal to zero), we obtain

$$\overline{d\chi/dt} = [3\mu n/c^2 a(1-e^2)^{1/2}] \times [2 - 2e^2 - (5 + 7e + 2e^2 + 3e^3 + 3e^4)/(1-e^2)^{3/2}]$$

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Reply by Author to H. R. Westerman

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The comment by H. R. Westerman is exemplary of a certain amount of confusion that has resulted from the introduction of the mean longitude L in the relativistic perturbations of Ref. 1. The purpose of this note is to clarify the meaning of the averaged rate in the mean longitude.

When the time rate of change of the mean anomaly phase angle χ is defined as in Eq. (4) of Ref. 1, then the mean anomaly rate is given by

$$dM/dt = n + (d\chi/dt) \tag{1}$$

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This expression has been discussed extensively (e.g., Ref. 4), and the interpretation of $d\chi/dt$ is that it describes the perturbative variation in the mean anomaly exclusive of the perturbations in the mean motion n. The form of $d\chi/dt$ as given by Eq. (4) of Ref. 1 is equivalent to the perturbative variation $\grave{M}$ of Herrick [cf. Eq. (190), Ref. 2]. Alternatively, it can be equated to the perturbative variation in the parameter $\epsilon - \tilde{\omega}$ of Plummer,[5] who also gives an expression for the total perturbative variation in the mean anomaly.

It follows from Eq. (1) and the definition of the mean longitude (i.e., $L = M + \omega + \Omega$) that

$$\frac{dL}{dt} = n + \frac{d\chi}{dt} + \frac{d\omega}{dt} + \frac{d\Omega}{dt} \tag{2}$$

where ω is the argument of the perifocus and Ω is the longitude of the ascending node. Therefore, the averaged motion in the mean longitude is given by

$$\overline{\frac{dL}{dt}} = \bar{n} + \overline{\frac{d\chi}{dt}} + \overline{\frac{d\omega}{dt}} + \overline{\frac{d\Omega}{dt}} \tag{3}$$

The only problem with this expression is in the interpretation of $\bar{n}$. The mean motion itself is given by an integral

$$n = n_0 + \int_{t_0}^{t} \left(\frac{dn}{dt}\right) dt \tag{4}$$

The average value of n with respect to the mean anomaly is, therefore, given by the following expression:

$$\bar{n} = n_0 + \frac{1}{2\pi} \int_0^{2\pi} \int_{t_0}^{t} \left(\frac{dn}{dt}\right) dt\, dM \tag{5}$$

The term to the right of the plus sign represents the deviation of the mean value of n from the initial value n_0 at the epoch t_0. Thus, if n_0 is used instead of $\bar{n}$ in the rate formula [Eq. (3)], than an additional term corresponding to the time integral of dn/dt enters in the expression for $\overline{dL}/dt$ or $\overline{dM}/dt$. For example, a formulation of this sort has been used by Edelbaum [Ref. 3, Eq. (21)].

Of course, when the periodic perturbations in L are required, it is necessary to include the extra term so that by Eqs. (2) and (4) the complete expression for the mean longitude is the following:

$$L - L_0 = n_0(t - t_0) + \int \int_{t_0}^{t} \left(\frac{dn}{dt}\right) dt^2 + \delta\chi + \delta\omega + \delta\Omega \tag{6}$$

where the symbol δ represents a variation from the unperturbed orbit with elements equal to the osculating elements at the epoch t_0.

Therefore,

$$\delta L = L - L_0 - n_0(t - t_0)$$

or, in agreement with Eq. (48) of Ref. 1, the perturbation in the mean longitude can be written in the following form:

$$\delta L = \int_{t_0}^{t} \delta n dt + \delta\chi + \delta\omega + \delta\Omega$$

Incidentally, we would like to take this opportunity to correct a typographical error in Eq. (46) of Ref. 1. That expression should read as follows:

$$\delta\omega = (\mu/2ac^2)[\tfrac{3}{2}{}^{3}\, e \sin M - (1 - \tfrac{2\,3}{2}\, e^2) \sin 2M - \tfrac{3}{2}\, e \sin 3M - \tfrac{7}{4}\, e^2 \sin 4M]$$

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Comment on "Vibration of a 45° Right Triangular Cantilever Plate by a Gridwork Method"

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It is of interest that Christensen[1] considers a sophisticated version of the Hardy-Cross method to be better than the classical Rayleigh-Ritz procedure. The authors of this comment contend that an elaborated Rayleigh-Ritz method must inevitably give better results. This conviction is sustained by the proof that a first-order error in modal shape estimate causes only a second-order error in frequency. To realize

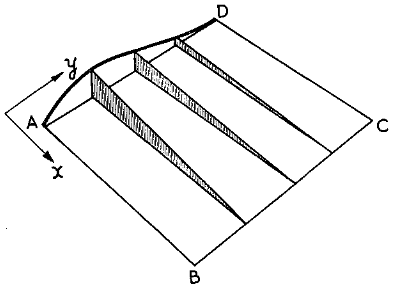

Fig. 1 Representation of displacement functions.

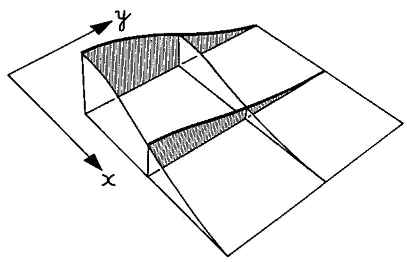

Fig. 2 Representation of displacement functions.

the practical advantage of this useful theoretical result in a complex engineering structure, it is necessary to eliminate first-order errors in structural idealization. Ideally the elements must have the same deflections at common boundaries and also the same slopes; further discussion on structural idealization can be found in Refs. 2 and 3.

The results of a recent investigation by the authors, considering a 3- × 4-in. rectangular cantilevered plate of uniform thickness analyzed into 1-in.-square elements, have been included in this comment. The nodes were allowed three degrees of freedom in the out-of-plane directions, giving 48 degrees of freedom to the plate.

The element displacements assumed for unit deflections are typified by the following expressions, applicable to a rectangle with vertices $\pm a$, $\pm b$:

$$w = \tfrac{1}{8}\, \{x + a\}\, \{x^2/a^2 - 1\}\, \{y/b + 1\}$$

$$w = \tfrac{1}{16}\, \{-x^3/a^3 + 3x/a + 2\}\, \{-y^3/b^3 + 3y/b + 2\}$$

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